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MATHEMATICS

Generalized Quadrangle *GQ*(4, 16) and Its Extensions

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We consider undirected graphs without loops or multiple edges. Given a vertex a in a graph Γ, let $\Gamma_i(a)$ denote the i-neighborhood of a, i.e., the subgraph induced by Γ on the set of all its vertices that are a distance of i away from a. Let $\Gamma(a) = \Gamma_1(a)$ and $a^\perp = \{a\} \cup \Gamma(a)$. If Γ is fixed, then, instead of $\Gamma(a)$, we write $[a]$.

Let F be a class of graphs. A graph Γ is said to be a locally F-graph if $[a]$ belongs to F for any vertex a of Γ. If F consists of the graphs isomorphic to a certain graph Δ, then Γ is called a locally Δ-graph.

Let Γ be a graph and a and b be two vertices from Γ. The number of vertices in $[a] \cap [b]$ is denoted by $\mu(a, b)$ if a and b are separated by a distance of 2 in Γ. The subgraph induced by $[a] \cap [b]$ is called a μ-subgraph.

The degree of a vertex is defined as the number of vertices in its neighborhood. Γ is called a regular graph of degree k if the degree of any vertex a in Γ is k. A graph Γ is called an edge-regular graph with parameters (v, k, λ) if Γ is a regular graph of degree k on v vertices in which each edge lies in λ triangles. Γ is called an amply regular graph with parameters (v, k, λ, μ) if Γ is an edge-regular graph with the corresponding parameters and $[a] \cap [b]$ contains μ vertices for any two vertices a and b separated by a distance of 2 in Γ. An amply regular graph is called strongly regular if its diameter is 2. Denote by $K_{m_1, \ldots, m_n}$ a complete multipartite graph $\{M_1, M_2, \ldots, M_n\}$ with parts M_i of orders m_i. If $m_1 = m_2 = \ldots = m_n = m$, then this graph is denoted by $K_{n \times m}$. The graph $K_{1, m}$ is called an m-claw.

An incidence system (X, B) with the set of points X and the set of blocks B is called a t-(v, k, λ)-scheme if $X = v$, each block is incident to exactly k points, and any t points are incident to exactly λ blocks. Any 2-scheme is a (v, b, r, k, λ)-scheme, where b is the number of blocks, each point is incident to r blocks, and it is true that $vr = bk$ and $(v - 1)\lambda = r(k - 1)$.

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The geometry G of rank 2 is an incidence system with a set of points P and a set of blocks B that does not have multiple blocks. Each block can be identified with the set of points that are incident to it, and the incidence becomes a usual inclusion. Two points from P are called collinear if they lie in a common block. The point graph of G is a graph on the set of points P in which two points are adjacent if they are distinct and collinear. The block graph is defined in a similar manner.

For $a \in P$, the residue G_a is defined as a geometry with the set P_a of points that are collinear to a and with the set of blocks $B_a = \{D - \{a\} | D$ is the block containing $a\}$ (in the case of a $t - (v, k, \lambda)$-scheme, the word "residue" is replaced by "derivative scheme"). If $a \in P$, $E \in B$, and $a \notin E$, then the pair (a, E) is called an antiflag. If any two blocks from B intersect in at most one point, then the set of blocks is called the set of straight lines L, while the geometry (P, L) is called a partial space of straight lines. A partial space of straight lines is of order (s, t) if each line contains exactly $s + 1$ points and each point lies on exactly $t + 1$ lines. A partial space of straight lines of order (s, t) is called a generalized quadrangle and is denoted by $GQ(s, t)$ if, for any antiflag (a, E), there is a unique straight line M containing a and intersecting E. The point graph of the geometry $GQ(s, t)$ is strongly regular with $v = (s + 1)(1 + st)$, $k = s(t + 1)$, $\lambda = s - 1$, and $\mu = t + 1$, and the geometry $GQ(s, t)$ can be recovered from its point graph.

Classical generalized quadrangles. Consider a nonsingular quadric Q with a Witt index of 2 for the projective space $PG(d, q)$, where $d = 3$, 4, or 5. Then the points of Q, together with the straight lines of Q, form a generalized quadrangle $Q(d, q)$. Consider a nonsingular Hermitian quadric H of the projective space $PG(d, q^2)$, $d = 3$ or 4. Then the points of H, together with the straight lines of H, form a generalized quadrangle $H(d, q^2)$; moreover, $H(3, q^2)$ has the parameters $s = q^2$ and $t = q$, while $H(4, q^2)$ has the parameters $s = q^2$ and $t = q^3$. The points of $PG(3, q)$, together with the straight lines that are totally isotropic with respect to symplectic polarity, form a generalized quadrangle $W(q)$ of order (q, q). If $d = 5$, the points and lines from Q form a generalized quadrangle $Q(5, q)$ of order (q, q^2) that is dual to $H(3, q^2)$.

The classification problem has been solved for $GQ(s, t)$ with $s \leq 3$ and for $GQ(4, t)$ with $t = 1, 2, 4$. Each of the existing generalized quadrangles $GQ(2, t)$, $t = 1, 2, 4$; $GQ(3, t)$, $t = 1, 3, 5, 9$; and $GQ(4, t)$, $t = 1, 2, 4$ is unique ($GQ(3, 3)$, up to duality). In this work, we prove the uniqueness of $GQ(4, 16)$.

Theorem 1. *Let a* 2-(16, 4, 3)-*scheme E be the extension of the unique generalized quadrangle* $GQ(2, 2)$ *as a* 1-*scheme. Then E cannot be extended.*

Corollary. *There exists a unique generalized quadrangle* $GQ(4, 16)$.

The classification of locally $GQ(s, t)$-graphs is a classical problem and has been solved for small s. For example, the classification of locally $GQ(3, t)$-graphs was completed in [1], while the amply regular locally $GQ(4, t)$-graphs were classified for $t = 2, 4, 6, 8$ in [2–5], respectively. In this paper, we study locally $GQ(4, 16)$-graphs.

Theorem 2. *There do not exist locally* $GQ(4, 16)$-*graphs.*

Let us prove Theorem 1. Let Γ be the point graph of $GQ(4, 16)$, u and w be two nonadjacent vertices from Γ, $X = [u] \cap [w]$, and $B = \Gamma_2(u) \cap \Gamma_2(w)$. Then $D = (X, B)$ is a 3-(17, 5, 3)-scheme, where a point is incident to a block if and only if they are adjacent in Γ.

Lemma 1. *Let E be a block and n_i be the number of blocks in B that intersect E at exactly i points. Then one of the following assertions holds:*

(1) *D is obtained from the unique* 3-(17, 5, 1)-*scheme by tripling each block, and Γ is the point graph of* $Q(5, 4)$.

(2) *For any block E, we have* $(n_0, n_1, n_2, n_3, n_4, n_5) = (28, 75, 80, 20, 0, 1)$ *and D is a double extension of the unique* $GQ(2, 2)$ *as a* 1-*scheme.*

Proof. All the assertions follow from Theorem 3.8 and Lemmas 3.3, 3.9 in [6].

Let a scheme $D = (X, B)$ be an extension of the 2-(16, 4, 3)-scheme that is a single-point extension of $GQ(2, 2)$ as a 1-scheme. Fix the point ∞ in X. Then (see [7, Example 9.7]) the derivative scheme $D_\infty = (X_\infty, B_\infty)$ has, as points, the antipodal classes $\{a, a^\sigma\}$ of the unique Taylor graph Δ, which is a locally $CQ(2,2)$-graph on 32 vertices, and, as blocks, $\{K, K^\sigma\}$, where K is a 4-clique from Δ and σ maps each vertex from Δ to its antipode. Thus, D has 60 blocks of the derivative scheme (containing the point ∞) and 144 external blocks. Furthermore, each 3-clique of Δ is contained in the unique block from B_∞ and in two external blocks, each edge of Δ is contained in 3 blocks from B_∞ and in 12 external blocks, and each vertex of Δ is contained in 15 blocks from B_∞ and in 45 external blocks. Therefore, the number of flags consisting of a 3-clique of Δ and the external block containing it is equal to 480.

Each pair of vertices u and w separated by distance 2 in Δ belongs to the subgraph $\{u, w^\sigma\} \cup \{u^\sigma, w\}$ (which is the union of isolated edges) and is contained in 12 external blocks. Therefore, $(X_\infty, B - B_\infty)$ is a 2-(16, 5, 12)-scheme. Note that an external block is a σ-admissible regular subgraph of Δ of degree 4 on 10 vertices.

Lemma 2. *Let E be an external block of the scheme D. Then the following assertions hold:*

(1) *E contains a* 3-*clique* $L = \{a, b, c\}$ *from* Δ, $E - (L \cup L^\sigma) = \{u, w\} \cup \{u, w\}^\sigma$ *is the union of isolated edges, and E contains either*

(i) *eight triangles* (*block of the first type*) *or*

(ii) *ten triangles* (*block of the second type*).

(2) *E does not contain* 3-*cliques from* Δ *and is obtained from the complete bipartite graph* $K_{5,5}$ *by deleting the maximal matching (block of the third type).*

(3) *If x_i is the number of blocks of the ith type, then* $x_1 = 60 - \frac{5x_2}{4}$ *and* $x_3 = 84 + \frac{x_2}{4}$.

Proof. Assume that E contains a 3-clique $L = \{a, b, c\}$ from Δ and $E - (L \cup L^\sigma) = \{u, w\} \cup \{u, w\}^\sigma$. Then any vertex from $L \cup L^\sigma$ is adjacent to exactly two vertices from $\{u, w\} \cup \{u, w\}^\sigma$ (altogether 12 edges). Therefore, each vertex from $\{u, w\} \cup \{u, w\}^\sigma$ is adjacent to exactly three vertices from $L \cup L^\sigma$ and $\{u, w\} \cup \{u, w\}^\sigma$ is the union of isolated edges. We can assume that $\Delta(u)$ contains a, b, c^σ. Then, without loss of generality, $\Delta(w)$ contains a, c, b^σ or a, b^σ, c^σ. Therefore, E contains eight triangles in the former case and ten triangles in the latter case. Assertion (1) is proved.

Suppose that E does not contain any 3-cliques from Δ. Then E is a triangle-free regular graph of degree 4 on 10 vertices and is obtained from the complete bipartite graph $K_{5,5}$ by deleting the maximal matching. Assertion (2) is proved.

If x_i is the number of blocks of the ith type, then $x_1 + x_2 + x_3 = 144$ and $8x_1 + 10x_2 = 480$. Therefore, $x_1 = 60 - \frac{5x_2}{4}$ and $x_3 = 144 - x_2 - \left(60 - \frac{5x_2}{4}\right) = 84 + \frac{x_2}{4}$. The lemma is proved.

Let us complete the proof of Theorem 1. Let a and c be adjacent vertices from Δ. Then $\Delta(a) \cap \Delta(c)$ is the union of three isolated edges and, by Lemma 2, the pairs a, c belongs to at least six external blocks containing 3-cliques from Δ. From this, $x_1 + x_2 \geq 72$ and $x_3 \leq 72$, a contradiction to the fact that, by Lemma 2, $x_3 \geq 84$. Theorem 1 is proved. The corollary follows from Theorem 1 and Lemma 1.

A subset of points Δ of a generalized quadrangle is called a hyperoval if any straight line intersects Δ at 0 or 2 points. In other words, a hyperoval in $GQ(s, t)$ is a triangle-free regular subgraph of degree $t + 1$ on an even number of vertices. It is well known that μ-subgraphs in locally $GQ(s, t)$-graphs are hyperovals.

Lemma 3. *Let the generalized quadrangle* $Q(5, 4)$ *of order* (4, 16) *contain a connected hyperoval* Δ. *Then the number of vertices in* Δ *is* 96 *or* 120.

Proof. The graph $GQ(4, 16)$ is constructed as one of rank 3 under the action of $PSU(4, 4).4$ (GAP includes this primitive representation). Then all orbits of 17-claws are found (1596 orbits). Finally, hyperovals are constructed using 17-claws as initial sets. It has been proved by computations in GAP that there are hyperovals only on 96 and 120 vertices.

Lemma 4. *There does not exist a connected locally $GQ(4, 16)$-graph.*

Proof. Let Γ be a connected amply regular locally $GQ(4, 16)$-graph. Let u be a vertex of Γ, x be the number of vertices from $\Gamma_2(u)$ adjacent to 96 vertices in $[u]$, and y be the number of vertices from $\Gamma_2(u)$ adjacent to 120 vertices in $[u]$. Then the number of edges between $[u]$ and $\Gamma_2(u)$ is $325 \cdot 256$ and is not divided by 3. On the other hand, this number is equal to $96x + 120y$, a contradiction. The lemma, together with Theorem 2, is proved.

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